10.5 Qualitative Theory

- Sometimes we can't easily solve a differential equation.
- But we can still sketch solution curves to understand <u>how</u> solutions behave

We will look at <u>autonomous</u> differential equations.

$$y' = g(y)$$

Derivative only depends on y, no t's involved.

Some important facts about autonomous D.E's

$$y' = g(y)$$

- If $g(y) = 0$ then $y' = 0$ so we have a constant solution, as t changes $g(y)$ stays 0 so $y = K \to$ some constant
- A non-constant solution must stay between constant solutions
- Each non-constant solution is either always increasing or always decreasing, $y' > 0$ then solution will have positive derivative.
- If a solution approaches a constant solution, it approaches asymptotically.

More important facts

- $y' > 0 \Rightarrow y$ is increasing

  $y' < 0 \Rightarrow y$ is decreasing

- $y'' > 0 \Rightarrow y$ concave up

  $y'' < 0 \Rightarrow y$ concave down

- If y' is at a max or min then $y'' = 0$

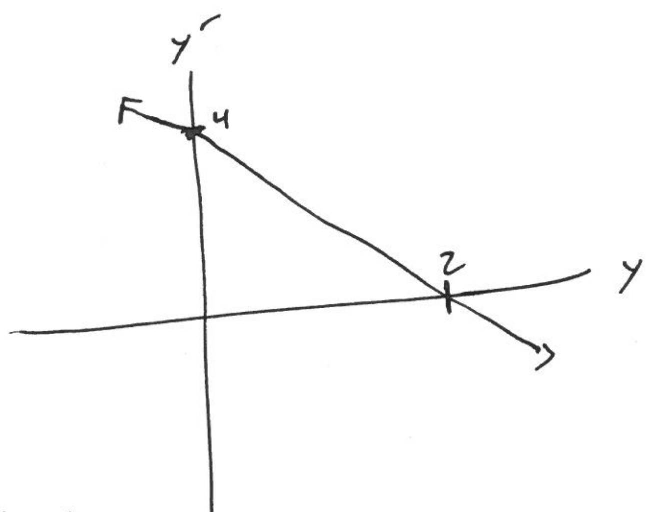
and if y' changes from increasing to decreasing then y'' changes from positive to negative. and we have a concavity change.

ex/ Sketch solutions ~~of~~ to

$y' = -2y + 4$ with initial conditions

$y(0) = -1$, $y(0) = 1$, $y(0) = 4$

① Sketch $g(y)$ graph, yy' ~~plane~~ plane



② • Look for constant solutions.

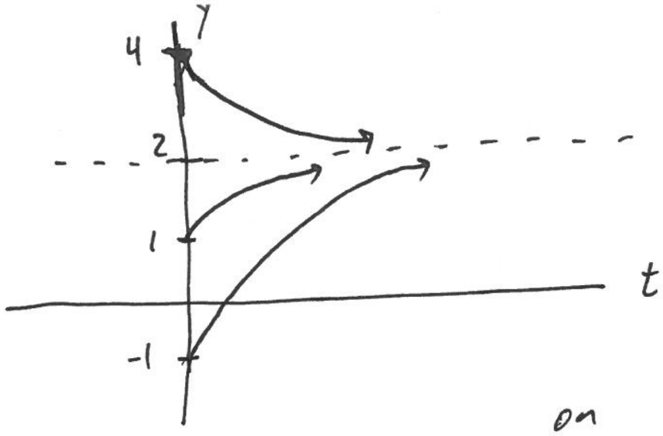
$y' = 0$ when $y = 2$

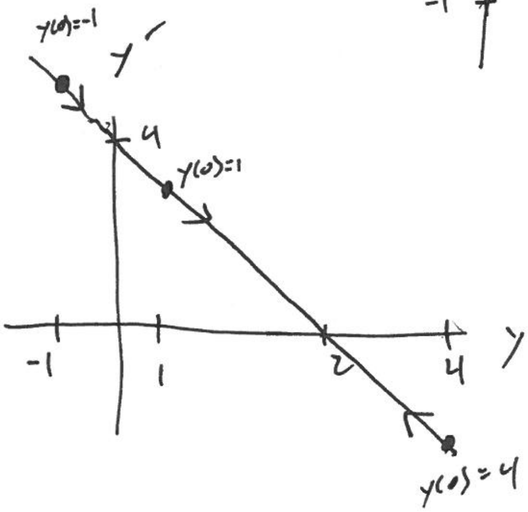
• Look for concavity changes (none here)

③ Set up ~~ty~~ t vs. y graph

Constant solutions will be given by horizontal line

④ Look at initial conditions on yy' graph to see behavior of solution (increasing / decreasing) and sketch





on yy' graph

$y(0)=-1$ $\quad y'>0 \Rightarrow$ y increasing

$y(0)=1$ $\quad y'>0 \Rightarrow$ y increasing

$y(0)=4$ $\quad y'<0 \Rightarrow$ y decreasing

for $y(0)=-1$
$y(0)=1$ since y is increasing we move to right on yy' graph so y' decreasing so concave down

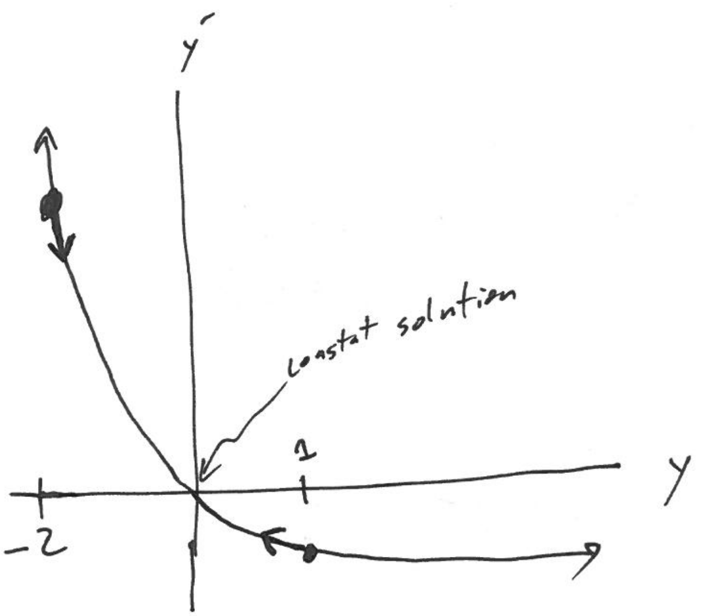
for $y(0)=4$ y is decreasing so on yy' graph we move to the left so y' increasing so concave up.

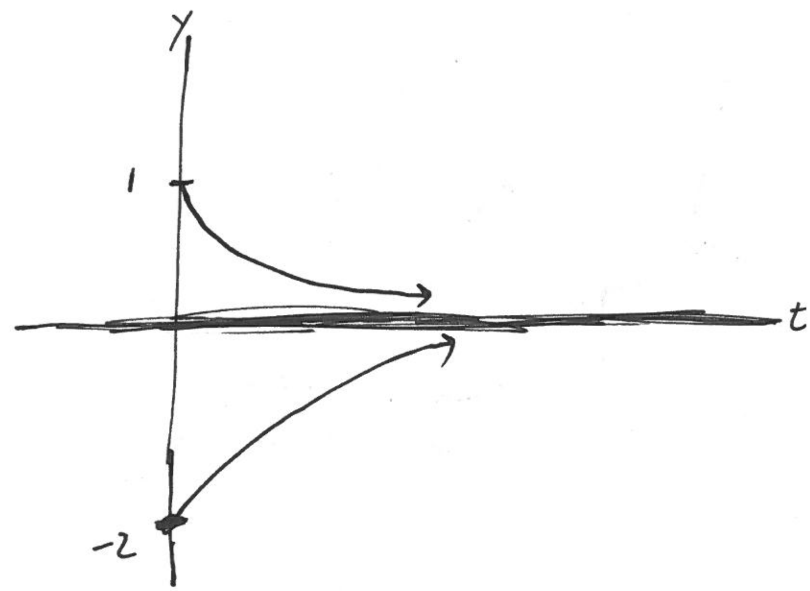
ex/ Sketch solutions to

$$y' = e^{-y} - 1 \quad \text{with initial conditions}$$

$$y(0) = -2$$

$$y(0) = 1$$





<u>$y(0) = -2$</u> when $y = -2$ $y' > 0$

so y is increasing

<u>$y(0) = 1$</u> when $y = 1$ $y' < 0$

so y is decreasing

what about

$$y' = e^{y} - 1$$

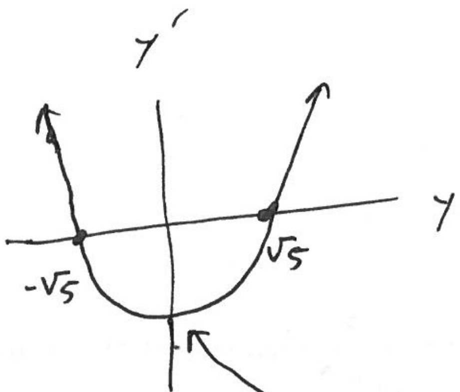
ex sketch solutions to $y' = y^2 - 5$

with initial conditions:

$y(0) = -4$

$y(0) = 2$

$y(0) = 3$



min at $y = 0$
(change inflection then)

$y' = 0$ when

$y^2 = 5 \Rightarrow y = \sqrt{5}$

$y = -\sqrt{5}$

$y(0) = -4$ then $y' > 0$

$y(0) = 2$ then $y' < 0$

$y(0) = 3$ then $y' > 0$